MATH20132 Calculus of Several Variable. 2019-20

Solutions to Problems 11 Extrema & Saddle Points

1. Suppose

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

is a matrix with real entries. Prove

i. If $\det M > 0$ (in particular $a \neq 0$) then

  a. M is positive definite if $a > 0$;

  b. M is negative definite if $a < 0$.

ii. If $\det M < 0$, then M is indefinite.

iii. If $\det M = 0$ then M is nondefinite.

Solution Consider, for $a \neq 0$, $\mathbf{x}^T M\mathbf{x}$ written as

$$\begin{aligned} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= ax^2 + 2bxy + cy^2 \\ &= a\left(x^2 + \frac{2b}{a}xy\right) + cy^2 \\ &= a\left(x + \frac{by}{a}\right)^2 - \frac{b^2y}{a} + \frac{acy^2}{a} \\ &= a\left(x + \frac{by}{a}\right)^2 + \frac{\det M}{a}y^2. \qquad (1) \end{aligned}$$

i. a. $\det M > 0$ and $a > 0$. Then the coefficients of both squares in (1) are positive so $\mathbf{x}^T M\mathbf{x} \geq 0$ for all $\mathbf{x}$, and is only 0 if $y = x = 0$. Thus M is positive definite.

i. b. $\det M > 0$ and $a < 0$. Then $(\det M)/a < 0$ so the coefficients of both squares in (1) are negative so $\mathbf{x}^T M\mathbf{x} \leq \mathbf{0}$ for all $\mathbf{x}$, and is only 0 if $y = x = 0$. Thus M is negative definite.

ii. If $\det M < 0$ then, whatever the sign of $a \neq 0$, the coefficients a and $(\det M)/a$ will be of *opposite* signs.

If $\mathbf{x}_1 = (1,0)^T$ then $\mathbf{x}_1^T M \mathbf{x}_1 = a$, while if $\mathbf{x}_2 = (-b/a, 1)^T$, $\mathbf{x}_2^T M \mathbf{x}_2 = (\det M)/a$. Thus $\mathbf{x}^T M \mathbf{x}$ takes both positive and negative values, i.e. it is indefinite.

iii. If $\det M = 0$ then $\mathbf{x}^T M \mathbf{x} = 0$ when $\mathbf{x} = (-b, a)^T \neq \mathbf{0}$ and so is nondefinite.

The above argument is based on $a \neq 0$ but if $\det M > 0$, i.e. $ac - b^2 > 0$ we must have $a \neq 0$. So the possibility of $a = 0$ only occurs when $\det M < 0$. If $a = 0$ complete the square for y, when, for $c \neq 0$,

$$\mathbf{x}^T M \mathbf{x} = c\left(y + \frac{bx}{c}\right)^2 - \frac{b^2}{c}x^2 = c\left(y + \frac{bx}{c}\right)^2 + \frac{\det M}{c}x^2.$$

Again, whatever the sign of $c \neq 0$, the signs of the coefficients of the squares are different and so $\mathbf{x}^T M \mathbf{x}$ is indefinite.

This leaves the case $a = c = 0$. But then the form is simply $-2bxy$ which takes positive and negative values. Hence, in this final case, the form is indefinite.

2. Find the critical points of the following functions.

i. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = x^3 + x - 4xy - 2y^2$;

ii. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = x(y+1) - x^2 y$;

iii. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = x^3 - 6xy + y^3$;

iv. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(\mathbf{x}) = x^4 + z^4 - 2x^2 + y^2 - 2z^2$;

v. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(\mathbf{x}) = x^2 + y^2 + z^2 + 2xyz$.

Use the Hessian matrix to determine whether each critical point is a local maximum, a local minimum or a saddle point.

Solution i. The critical points simultaneously satisfy $\nabla f(\mathbf{x}) = \mathbf{0}$ which in component form becomes

$$3x^2 + 1 - 4y = 0 \quad \text{and} \quad -4x - 4y = 0.$$

From the second $y = -x$, which in the first gives $3x^2 + 4x + 1 = 0$. Thus $x = -1$ or $-1/3$. These give the critical points $(-1, 1)^T$ and $(-1/3, 1/3)^T$.

The Hessian matrix is

$$Hf(\mathbf{x}) = \begin{pmatrix} 6x & -4 \\ -4 & -4 \end{pmatrix}.$$

At $\mathbf{x}_1 = (-1, 1)^T$ this gives

$$Hf(\mathbf{x}_1) = \begin{pmatrix} -6 & -4 \\ -4 & -4 \end{pmatrix}$$

which has determinant 8 and so, since $a_{11} < 0$, the matrix is negative definite and f has a local maximum.

At $\mathbf{x}_2 = (-1/3, 1/3)^T$ this gives

$$Hf(\mathbf{x}_2) = \begin{pmatrix} -2 & -4 \\ -4 & -4 \end{pmatrix}$$

which has determinant -8. Therefore the matrix is indefinite and f has a saddle at $\mathbf{x}_2$.

ii. Critical points satisfy

$$y + 1 - x^2 = 0 \quad \text{and} \quad x - x^2 = 0.$$

From the second, $x = 0$ or 1. If $x = 0$ the first gives $y = -1$. If $x = 1$ the first gives $y = 1$. So the two critical points are $\mathbf{x}_1 = (0, -1)^T$ and $\mathbf{x}_2 = (1, 1)^T$.

The Hessian matrix is

$$Hf(\mathbf{x}) = \begin{pmatrix} -2y & 1 - 2x \\ 1 - 2x & 0 \end{pmatrix}.$$

The determinant is -1 at both critical points and so they are both saddle points.

iii. Critical points satisfy

$$3x^2 - 6y = 0 \quad \text{and} \quad -6x + 3y^2 = 0.$$

From the first equation $y = x^2/2$. In the second this values of y gives $2x = (x^2/2)^2$, i.e. $x^4 = 8x$. This means either $x = 0$ or $x = 2$. So the two critical points are $\mathbf{x}_1 = (0, 0)^T$ and $\mathbf{x}_2 = (2, 2)^T$.

The Hessian matrix is

$$Hf(\mathbf{x}) = \begin{pmatrix} 6x & -6 \\ -6 & 6y \end{pmatrix}.$$

Then $\det Hf(\mathbf{x}_1) = -36 < 0$ so $\mathbf{x}_1$ is a saddle point.

Also $\det Hf(\mathbf{x}_2) = 108 > 0$ with $a_{11} = 12 > 0$ and so $\mathbf{x}_2$ is a local minimum.

iv. Critical points satisfy

$$4x^3 - 4x = 0, \quad d_2 f(\mathbf{x}) = 2y = 0 \quad \text{and} \quad 4z^3 - 4z = 0.$$

Thus $y = 0$, $x = 0$ or ± 1 and $z = 0$ or ± 1. This gives 9 critical points.

The Hessian matrix is

$$Hf(\mathbf{x}) = \begin{pmatrix} 12x^2 - 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12z^2 - 4 \end{pmatrix}.$$

That the non-zero entries only lie on the diagonal simplifies the problem.

At $(0,0,0)^T$, the Hessian matrix is

$$\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

With entries of different sign we have a saddle point.

At $(\pm 1, 0, 0)^T$, the Hessian matrix is

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Again we have a saddle point. Similarly at $(0,0,\pm 1)^T$ we will have a saddle point.

In the remaining four cases $(\pm 1, 0, \pm 1)^T$ the Hessian matrix is

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

With all entries positive we have local minima at these four points.

v. Critical points satisfy

$$2x + 2yz = 0, \quad 2y + 2xz = 0 \quad \text{and} \quad 2z + 2xy = 0.$$

That is $x + yz = 0$, $y + xz = 0$ and $z + xy = 0$.

Substitute the first into the third, so $z - y^2 z = 0$. Thus, either $z = 0$ or $y = 1$ or $y = -1$.

If $z = 0$ then $x = y = 0$.

If $y = 1$ then $x + z = 0$ and $xz = -1$. This has two solutions $(x, z) = (1, -1)$ or $(-1, 1)$.

If $y = -1$ then $x - z = 0$ and $xz = 1$. This has two solutions $(x, z) = (1, 1)$ or $(-1, -1)$.

Hence we have found 5 critical points $(0, 0, 0)^T$, $(1, 1, -1)^T$, $(-1, 1, 1)^T$, $(1, -1, 1)^T$ and $(-1, -1, -1)^T$.

The Hessian matrix is

$$Hf(\mathbf{x}) = \begin{pmatrix} 2 & 2z & 2y \\ 2z & 2 & 2x \\ 2y & 2x & 2 \end{pmatrix}.$$

At $(0, 0, 0)^T$ the matrix is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

With positive entries the point is a local minimum.

At $(1, 1, -1)^T$ the matrix is

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Calculating the determinants of the principle minors $\det A_1 = 2$, $\det A_2 = 0$ and $\det A_3 = -32$. Because of the 0 for one of these determinants the point is a saddle point.

In fact, for all the points $(1, 1, -1)^T$, $(-1, 1, 1)^T$, $(1, -1, 1)^T$ and $(-1, -1, -1)^T$ we have $z = \pm 1$ so $\det A_2 = 0$ for the Hessian matrices for each point. Hence all the points are saddle points.